HEAT EXCHANGE IN THE FLOW OF NON-NEWTONIAN FLUIDS

P. V. Tsoi and Sh. Nuriddinov UDC 532.517.2:532.135

A method is proposed for solution of convective heat-exchange problems for the flow of non-Newtonian fluids in pipes and channels. Problems with a linear rise in the tube wall temperature are investigated in detail.

Various dependences between the stress and shear rate are known for non-Newtonian fluid flows [1]. The most widespread and simple is the power-law rheological flow

$$\tau = -k\left(\frac{dw}{dr}\right)^m. \tag{1}$$

The velocity profiles for laminar hydrodynamically stabilized flows of such media are determined by the equations

$$w(r) = \frac{(3m+1)w_{av}}{m+1}\left[1-\left(\frac{r}{R}\right)^{\frac{m+1}{m}}\right] \tag{2}$$

in a circular tube and

$$w(y) = \frac{(2m+1)w_{av}}{m+1}\left[1-\left(\frac{y}{b}\right)^{\frac{m+1}{m}}\right], \quad -b\leqslant y\leqslant b \tag{3}$$

in a plane-parallel channel.

Results of investigations on heat exchange are presented in [2] by the method of the integral heat balance far from the entrance to the tube for the flow of anomalous fluids with a rheological power law. The authors show that the temperature curves differ only slightly for diverse values of m for substantially different velocity profiles. Such a result is evidently valid just for the case when the heat of friction is not taken into the computation.

In order to determine the temperature in a fluid flow in a circular tube, and the length of the initial section of thermal stabilization behind which the solutions in [2] will be valid, the following problem must be solved:

$$\frac{(3m+1)}{m+1}\left(1-\rho^{\frac{m+1}{m}}\right)\frac{\partial T}{\partial X} = \frac{\partial^2 T}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial T}{\partial \rho}+\dot{\Theta}(\rho,\ X),\ X=\frac{1}{\mathrm{Pe}}\cdot\frac{z}{R}, \tag{4}$$

$$[T(\rho,\ X)]_{X=0}=T_0,\ [T(\rho,\ X)]_{\rho=1}=\varphi(X),\ 0\leqslant z<\infty. \tag{5}$$

We obtain the solution to the problem by the method elucidated in [6, 7]. Let us assume

$$\overline{T}(\rho,\ s) = \int_0^\infty T(\rho,\ X)\exp(-sX)\,dX;$$

then after using the Laplace transform, the problem is reduced to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\overline{T}}{d\rho}\right)-\frac{(3m+1)}{m+1}\left(1-\rho^{\frac{m+1}{m}}\right)\left[s\overline{T}(\rho,\ s)-T_0\right]=0, \tag{6}$$

$$[\overline{T}(\rho,\ s)]_{\rho=1}=\overline{\varphi}(s). \tag{7}$$

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Following the method of Bubnov–Galerkin [5], let us seek the solution of the boundary value problem (6), (7) as

$$\overline{T}_n(\rho,\ s) = \overline{\varphi}(s) + \sum_{k=1}^{n} \overline{a}_k(s)\,\psi_k(\rho), \tag{8}$$

where the coordinate functions $\psi_k(\rho)$ (k = 1, 2, . . . , n) are linearly independent and satisfy the homogeneous boundary conditions

$$[\psi_k(\rho)]_{\rho=1} = 0.$$

The transform coefficients $\overline{a}_k(s)$ (k = 1, 2, . . . , n) for which (8) best satisfies the boundary value problem (6), (7) are determined from the system [7]

$$\left\{\sum_{k=1}^{n} (A_{jk} + B_{jk}s)\,\overline{a}_k(s) = D_j(s)\right\} \quad (j = 1,\ 2,\ \ldots,\ n), \tag{9}$$

where

$$A_{jk} = A_{kj} = \int_0^1 \frac{d\psi_j}{d\rho}\cdot\frac{d\psi_k}{d\rho}\cdot\rho d\rho,$$

$$B_{jk} = B_{kj} = \frac{3m+1}{m+1}\int_0^1 \left(1-\rho^{\frac{m+1}{m}}\right)\psi_j\psi_k\cdot\rho d\rho; \tag{10}$$

$$D_j(s) = \frac{3m+1}{m+1}\left[T_0 - s\overline{\varphi}(s)\right]\int_0^1 \left(1-\rho^{\frac{m+1}{m}}\right)\psi_j(\rho)\,\rho d\rho + \int_0^1 \overline{\dot{\Theta}}\rho\psi_j d\rho.$$

Let

$$\overline{a}_k(s) = \frac{\left[T_0 - s\overline{\varphi}(s)\right]\Delta_k(s)}{\Delta(s)} \tag{11}$$

be the solution of the governing Bubnov–Galerkin system for $\dot{\Theta}(\rho,\ x) = 0$; $\Delta(s) = |A + Bs|$ is the main determinant of the system (9). Going over to the domain of originals formally, we obtain the solution of the problem posed as

$$T_n(\rho,\ X) = \varphi(X) + \sum_{k=1}^{n} a_k(X)\,\psi_k(\rho), \tag{12}$$

where

$$a_k(X) = \sum_{i=1}^{n}\int_0^X \frac{\Delta_k(s_i)}{\Delta'(s_i)}\exp\left[s_i(X-\alpha)\right]\varphi^*(\alpha)\,d\alpha;$$

$\varphi^*(X) \doteqdot T_0 - s\overline{\varphi}(s)$; s_i are the roots of the equation $\Delta(s) = 0$. The matrix of the main determinant of the system (9) is symmetric, and its elements are positive. All the roots s_i are hence real and negative.

Let us consider the problem for a linear rise in wall temperature

$$[T(\rho,\ X)]_{\rho=1} = T_0 + \Delta Tz = T_0 + \Delta T^*X, \quad \Delta T^* = \Delta T\,\mathrm{Pe}\,R. \tag{13}$$

Let us first investigate the temperature field far from the tube entrance. We seek this solution in the form

$$T(\rho,\ X) = T_0 + \Delta T^*X + T_1(\rho), \tag{14}$$

where the unknown function $T_1(\rho)$ satisfies the condition

$$T_1(1) = 0, \quad \left(\frac{dT_1}{d\rho}\right)_{\rho=0} = 0. \tag{15}$$

Let us substitute (14) into (4) and let us assume $\dot{\Theta}(\rho, X) = 0$, then

$$\frac{(3m+1)}{m+1}\left(1-\rho^{\frac{m+1}{m}}\right)\Delta T^* = \frac{1}{\rho}\cdot\frac{d}{d\rho}\left(\rho\frac{dT_1}{d\rho}\right). \tag{16}$$

Integrating this equation under the conditions (15), we obtain

$$T_1(\rho) = -\frac{(3m+1)\Delta T^*}{m+1}\left[\frac{5m^2+6m+1}{4(3m+1)^2} - \frac{\rho^2}{4} + \left(\frac{m}{3m+1}\right)^2 \rho^{\frac{3m+1}{m}}\right]. \tag{17}$$

The relative excess temperature is reduced to

$$\Theta(\rho, X, m) = \frac{T-T_0}{\Delta T^*} = X - \frac{3m+1}{m+1}\left[\frac{5m^2+6m+1}{4(3m+1)^2} - \frac{\rho^2}{4} + \left(\frac{m}{3m+1}\right)^2 \rho^{\frac{3m+1}{m}}\right], \tag{18}$$

from which

$$-\left(\frac{d\Theta}{d\rho}\right)_{\rho=1} = \frac{1}{2}, \tag{19}$$

i.e., the temperature gradient (heat flux) at the wall is independent of the coordinate X and the rheological parameter m for a linear rise in the wall temperature, and equals a constant.

The minimum Nusselt criterion is

$$\mathrm{Nu}_{\min} = N(m) = \frac{-2\left(\frac{\partial\Theta}{\partial\rho}\right)_{\rho=1}}{\bar{\Theta} - \Theta_{\text{ст}}} = \frac{8(5m+1)(3m+1)(m+1)}{31m^3+43m^2+13m+1}. \tag{20}$$

For a Newtonian fluid (m = 1) we obtain

$$\Theta(\rho, X, 1) = X - \left(\frac{3}{8} - \frac{1}{2}\rho^2 + \frac{1}{8}\rho^4\right), \quad \mathrm{Nu}_{\min} = \frac{48}{11} = 4.364. \tag{21}$$

Assuming m = 0 we obtain for channel flow

$$\Theta(\rho, X, 0) = X - \frac{1}{4}(1-\rho^2), \quad \mathrm{Nu}_{\min} = 8. \tag{22}$$

Therefore, for the two particular cases we have obtained results agreeing with those of Ferguson [4] and Ustimenko et al. [3]. The dependence of $\mathrm{Nu}_{\min}$ on the parameter m, computed by means of (20), is represented graphically in Fig. 1.

For an ideally dilatant material, we obtain from (20) for $m \to \infty$

$$\lim N(m) = \frac{120}{31} = 3.871.$$

The velocity profiles and temperatures in a non-Newtonian fluid stream are represented in Fig. 2 for different values of the parameter m on the stabilized sections.

Let us find the temperature in a stream of medium and the regularities of the heat exchange at the initial section of the tube. Our preliminary studies of the temperature far from the tube entrance permit the construction of the solution in that system of coordinate functions for which we obtain the best approximation. Let us seek the approximate solution in a family of functions of the form

$$\bar{T}_n(\rho, s) = \frac{T_0}{s} + \frac{\Delta T^*}{s^2} + \sum_{k=1}^{n}\bar{a}_k(s)\left[\frac{5m^2+6m+1}{4(3m+1)^2} - \frac{\rho^2}{4} + \left(\frac{m}{3m+1}\right)^2\rho^{\frac{3m+1}{m}}\right]\rho^{2(k-1)}. \tag{23}$$

For simplicity, all the subsequent calculations will be carried out for specific values of the parameter m.

For m = 1/3 (pseudoplastic) the governing system (9) reduces for the third approximation (n = 3) to the following:

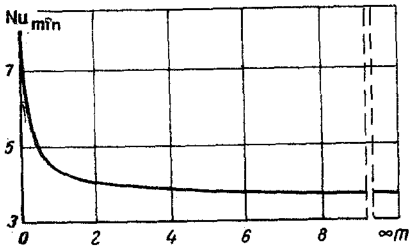

Fig. 1

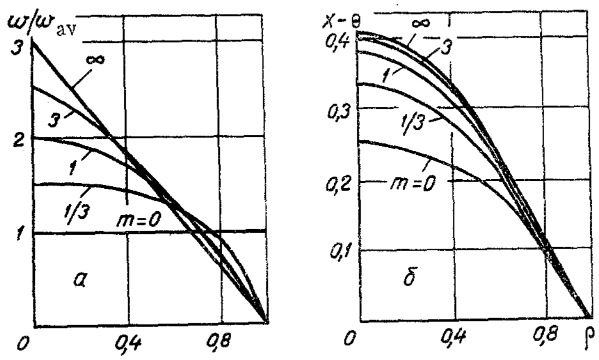

Fig. 2

Fig. 1. Dependence of Nu_{min} on the parameter m.

Fig. 2. Velocity profiles (a) and temperatures (b) in a stream on non-Newtonian fluid for m = 0; 1/3; 1; 3; ∞.

$$\left(\frac{19}{432}+\frac{701}{68040}s\right)\bar{a}_1(s)+\left(\frac{1}{84}+\frac{349}{161280}s\right)\bar{a}_2(s)+\left(\frac{43}{8640}+\frac{2213}{2993760}s\right)\bar{a}_3(s)=-\frac{19}{288}\frac{\Delta T^*}{s},$$

$$\left(\frac{1}{84}+\frac{349}{161280}s\right)\bar{a}_1(s)+\left(\frac{13}{1080}+\frac{2219}{2993760}s\right)\bar{a}_2(s)+\left(\frac{809}{102060}+\frac{1397}{4354560}s\right)\bar{a}_3(s)=-\frac{1}{56}\frac{\Delta T^*}{s},$$

$$\left(\frac{43}{8640}+\frac{2213}{2993760}s\right)\bar{a}_1(s)+\left(\frac{809}{102060}+\frac{1397}{4354560}s\right)\bar{a}_2(s)+\left(\frac{67}{10080}+\frac{349}{2162160}s\right)\bar{a}_3(s)=-\frac{3}{640}\frac{\Delta T^*}{s}, \quad (24)$$

from which we obtain by solving a truncated first-order system

$$\bar{a}_1(s)=-\frac{3\Delta T^*}{2}\left[\frac{1}{s}-\frac{1}{s+4.2689}\right].$$

The relative excess temperature is written in a first approximation as

$$\Theta(\rho,\ X)=\frac{T-T_0}{\Delta T^*}=X-\frac{3}{2}\left[1-\exp(-4.2689X)\right]\left(\frac{2}{9}-\frac{\rho^2}{4}+\frac{1}{36}\rho^6\right). \quad (25)$$

We determine $\bar{a}_1(s)$, $\bar{a}_2(s)$ from the truncated second-order Bubnov–Galerkin system, and then find the coefficient-originals $\bar{a}_1(X)$, $\bar{a}_2(X)$. Then the solution in a second approximation is

$$\Theta(\rho,\ X)=\frac{T-T_0}{\Delta T^*}=X-\left[0.0417-0.0445\exp(-4.1802X)+0.0028\right.$$

$$\left.\times\exp(-31.6180X)\right](8-9\rho^2+\rho^6)-0.0142\left[\exp(-4.1802X)-\exp(-31.6180X)\right](8\rho^2-9\rho^4+\rho^6). \quad (26)$$

For the mean mass temperature in the fluid stream we obtain

$$\bar{\Theta}(X)=X-0.1981+0.1931\exp(-4.1802X)+0.0050\exp(-31.6180X). \quad (27)$$

Let us write the local Nusselt criterion by referring the coefficient of heat exchange to the local temperature difference $\Theta_W-\bar{\Theta}$:

$$\mathrm{Nu}=N(X)=\frac{2}{\Theta_W-\bar{\Theta}}\cdot\left(\frac{\partial\Theta}{\partial\rho}\right)_{\rho=1}=\frac{5.053\left[1-0.727\exp(-4.180X)-0.273\exp(-31.618X)\right]}{1-0.975\exp(-4.180X)-0.025\exp(-31.618X)}. \quad (28)$$

Behind the thermal stabilization section, we obtain from (25), (26), (28) for sufficiently large X

$$\Theta(\rho,\ X)=X-\frac{3}{2}\left(\frac{2}{9}-\frac{\rho^2}{4}+\frac{1}{36}\rho^6\right),\quad \lim_{X\to\infty}N(X)=5.053. \quad (29)$$

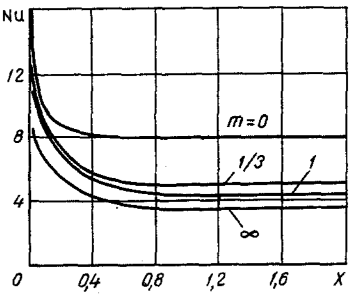

Fig. 3. Change in the local Nusselt criterion for a linear rise in wall temperature and n = 0; 1/3; 1; ∞.

It is easy to note that the results obtained agree with the values of (18), (20) for m = 1/3. Therefore, the approximate solutions far from the tube entrance agree with the exact solutions.

It follows from (25), (26) that the discrepancy in the calculation of the first eigenvalue between the first and second approximations is about 2%. Equating the main determinant of the system (24) to zero, we obtain

$$s_1 = -4.1742; \quad s_2 = -24.8021; \quad s_3 = -123.6350.$$

We see that the difference in calculating the first eigenvalue between the second and third approximations is 0.14%. Hence, it can be considered that the first eigenvalue has been determined with sufficient accuracy. Therefore, the temperature stabilization along a fluid stream, computed by means of the approximate solution (26), will practically agree with the exact value.

For channel flow (m = 0, $w(\rho) = w_{av}$ = const) the relative excess temperature and local Nusselt criterion are written in a second approximation as

$$\Theta(\rho,\ X) = \frac{T - T_0}{\Delta T^*} = X - \frac{1}{4}\,[1 - 1.1002\exp(-5.7842X) + 0.1002$$

$$\times \exp(-36.8824X)]\,(1-\rho^2) - 0.1072\,[\exp(-5.7842X) - \exp(-36.8824X)]\,(\rho^2 - \rho^4), \tag{30}$$

$$\mathrm{Nu} = N(X) = \frac{8\,[1 - 0.6715\exp(-5.7842X) - 0.3285\exp(-36.8824X)]}{1 - 0.9568\exp(-5.7842X) - 0.0432\exp(-36.8824X)}, \tag{31}$$

from which we obtain relationships agreeing with (18), (20) for m = 0 in the limit as $X \to \infty$.

The first eigenvalue in the approximate solution (30) equals 5.7842 and differs from the exact value $\varepsilon_1^2 = 5.7831$, where ε_1 is the first root of the zero-order Bessel function of the first series, by just 0.019%. Therefore, the exponential temperature stabilization along a stream, computed by means of (30), will agree with the temperature stabilization in the exact solution. Let us note that (4) for channel flow (m = 0) agrees with the equation of heat conduction. Hence, there is an opportunity to compare the approximate and exact solutions. Such comparisons have shown that the solutions in a second and third approximation permit carrying out highly accurate thermal engineering computations.

For an ideally dilatant medium (m = ∞) the temperature distribution in the stream for a linear rise in the wall temperature reduces in a third approximation to the form

$$\Theta(\rho,\ X) = \frac{T - T_0}{\Delta T^*} = X - 0.0844\,(5 - 9\rho^2 + 4\rho^3) + \sum_{k=1}^{3} \psi_k^*(\rho)\exp(-s_k X), \tag{32}$$

where

$$\begin{aligned} \psi_1^*(\rho) &= (3.2484 - 1.6105\rho^2 + 0.9301\rho^4)\,\psi_0(\rho);\\ \psi_2^*(\rho) &= (-0.2254 + 1.8831\rho^2 - 1.5339\rho^4)\,\psi_0(\rho);\\ \psi_3^*(\rho) &= (0.0159 - 0.2726\rho^2 + 0.6058\rho^4)\,\psi_0(\rho);\\ \psi_0(\rho) &= \frac{1}{36}\,(5 - 9\rho^2 + 4\rho^3); \quad s_1 = 3.2641; \quad s_2 = 21.9749; \quad s_3 = 141.3009. \end{aligned} \tag{33}$$

The local Nusselt criterion is

$$\mathrm{Nu} = \frac{-2\left(\dfrac{\partial\Theta}{\partial\rho}\right)_{\rho=1}}{\bar{\Theta} - \Theta_{\text{ст}}} = \frac{12\cdot 0.0844 - 2\displaystyle\sum_{k=1}^{3}\left(\frac{\partial\psi_k^*}{\partial\rho}\right)_{\rho=1}\exp(-s_k X)}{3.1\cdot 0.0844 - 6\displaystyle\sum_{k=1}^{3}\exp(-s_k X)\int_0^1 \psi_k^*(\rho)\,(\rho - \rho^2)\,d\rho}, \tag{34}$$

from which

$$\lim \mathrm{Nu} = \frac{12}{3.1} = 3.871.$$

Finally, let us consider the heat exchange in a flow of normal Newtonian fluid (m = 1). Let us put m = 1 in (2) and let us solve the boundary value problem (4), (5) for a linear rise in the wall temperature (13).

For n = 3 the system (9) reduces to

$$\left(\frac{11}{384}+\frac{59}{7680}s\right)\bar{a}_1(s)+\left(\frac{13}{1920}+\frac{5}{3584}s\right)\bar{a}_2(s)+\left(\frac{1}{384}+\frac{31}{71680}s\right)\bar{a}_3(s)=\frac{11}{192}\left[T_0-s\bar{\varphi}(s)\right],$$

$$\left(\frac{13}{1920}+\frac{5}{3584}s\right)\bar{a}_1(s)+\left(\frac{9}{1280}+\frac{31}{71680}s\right)\bar{a}_2(s)+\left(\frac{61}{13440}+\frac{113}{645120}s\right)\bar{a}_3(s)=\frac{13}{960}\left[T_0-s\bar{\varphi}(s)\right], \quad (35)$$

$$\left(\frac{1}{384}+\frac{31}{71680}s\right)\bar{a}_1(s)+\left(\frac{61}{13440}+\frac{113}{645120}s\right)\bar{a}_2(s)+\left(\frac{17}{4480}+\frac{3}{35840}s\right)\bar{a}_3(s)=\frac{1}{192}\left[T_0-s\bar{\varphi}(s)\right].$$

In our case $T_0 - s\bar{\varphi}(s) = -\Delta T^*/s$. Determining the coefficients $\bar{a}_k(s)$ (k = 1, 2, 3) from this system and returning to the domain of originals, we can find the approximate solution. We omit all intermediate calculations and present the final results of a temperature computation in the third approximation

$$\Theta(\rho, X) = X - \frac{1}{8}(3-4\rho^2+\rho^4)+\frac{1}{16}\sum_{k=1}^{3}\psi_k^*(\rho)\exp(-s_k X), \quad (36)$$

where

$$\psi_1^* = 8.2392 - 14.0940\rho^2 + 8.4710\rho^4 - 3.1430\rho^6 + 0.5268\rho^8;$$
$$\psi_2^* = -2.5883 + 7.1988\rho^2 - 8.7904\rho^4 + 5.1568\rho^6 - 0.9769\rho^8;$$
$$\psi_3^* = 0.3491 - 1.1048\rho^2 + 2.3194\rho^4 - 2.0138\rho^6 + 0.4501\rho^8;$$
$$s_1 = 3.6572; \quad s_2 = 23.1112; \quad s_3 = 137.6855.$$

Let us note that to conserve the stability of the governing system (35), all the intermediate calculations have been performed to the accuracy of the sixth place after the decimal.

For X = 0, $\rho = 1$, respectively, we obtain from (36)

$$T(\rho, 0) = T_0, \quad T(1, X) = T_0 + \Delta T^* X,$$

i.e., the approximate solution satisfies the temperature conditions at the tube entrance and walls completely.

For the local Nusselt criterion we obtain

$$\mathrm{Nu} = \frac{1+\frac{1}{16}\sum_{k=1}^{3}\left(\frac{\partial\psi_k^*}{\partial\rho}\right)_{\rho=1}\exp(-s_k X)}{\frac{11}{48}+\frac{1}{4}\sum_{k=1}^{3}\exp(-s_k X)\int_0^1\psi_k^*(\rho)(\rho-\rho^3)\,d\rho}, \quad (37)$$

from which

$$\lim \mathrm{Nu} = \frac{48}{11} = 4.364. \quad (38)$$

Let us note that the system (35) permits investigation of the heat exchange for a Newtonian fluid for any other laws of wall temperature variation.

The change in the local Nusselt criterion along a flow of a medium is represented graphically in Fig. 3 for m = 0; 1/3; 1; ∞.

The results of investigations show that the exponential drop in the local Nusselt criterion at a tube entrance depends on the parameter m and the length of the thermal stabilization section depends on the velocity profile. For an equivalent value of Pe the thermal stabilization section takes on a minimal value for channel flow and a maximal value for an ideally dilatant medium ($m = \infty$).

In conclusion, let us note that the heat exchange in a flow of anomalous media in a plane channel taking account of the heat of friction and other internal sources of heat liberation is investigated completely analogously.

NOTATION

$T(\rho, X)$ is the temperature in the fluid stream;
$\bar{T}(\rho, s)$ is the Laplace transform of the temperature;
r, R, ρ are the running radius, radius, and dimensionless radius of the tube;
$\doteqdot$ is the sign of transformation from original to transform and back;
s is the parameter of the integral Laplace transform;
w_{av} is the mean stream velocity.

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